ALGEBRA THROUGH THE LENS OF FUNCTIONS

JUNIOR CERTIFICATE HIGHER LEVEL

PART 2 OF 2

Introduction

To Teachers

Algebra Through the Lens of Functions Part 1 and 2 have been designed by the Maths Development Team for use by teachers of mathematics. *Algebra Through the Lens of Functions* Part 1 of 2 should be used prior to engaging with Part 2. Both part 1 and part 2 treat the art of teaching algebra through the lens of functions through a series of units. The material contained in the document is suitable for all levels and abilities but is particularly suited to Junior Certificate Higher Level. It includes a discussion of activities, tasks and the formation of connections suitable for classroom use. When necessary, required subject matter content is covered as well.

Both Part 1 and 2 were written in response to the many teachers who attended continuing professional development courses given by the authors and were unable to find material in a single convenient source. The authors collaborated with the Maths Inspectorate of the Department of Education & Skills to provide a collection of activities and strategies for Junior Certificate mathematics classes.

Activity Book

Visualising Patterns for Linear Relationships and *Visualising Patterns for Quadratic Relationships* are activity workbooks which can be used to supplement the text in Part 1 and Part 2 respectively.

Visualising Patterns for Quadratic Relationships contains fourteen patterns that may be used to extend the ideas presented by the corresponding units in this text.

Visualising Patterns for Quadratic Relationships can be downloaded from here. Larger versions of the images are available in PDF form here and as an interactive PowerPoint here.

Organisation, Format & Special Features of the Units

Linear Relationships

Units 1 to 10 are contained in *Algebra Through the Lens of Functions* Part 1 of 2. This resource can be found here.

Quadratic Relationships

Units 11 to 22, in this document: *Algebra Through the Lens of Functions* Part 1 of 2, deal with quadratic relationships.

Unit 11 includes forming quadratic expressions from visual patterns, an approach similar to that adopted in *Algebra Through the Lens of Functions* Part 1 of 2. This Unit also allows for the skills of simplifying expressions into their equivalent forms and substitution into expressions to be explored. More importantly, two questions that will involve the need to generalise are introduced. These questions are intended to motivate the content of the next few Units and consolidate the notion that the use of algebra generally is the more efficient strategy, when dealing with increasingly complex problems.

In Unit 12, geometric shapes are utilised in visualising like and unlike terms in quadratic and linear expressions while simplifying the sum and difference of like terms and simple factors by utilising rectangular shapes are dealt with in Unit 13. Unit 14 deals with multiplying expressions and factorising by grouping and helps students recognise that one advantage of factorising by grouping is that the area of rectangular shapes can be expressed concisely. Unit 15 deals with algebraic multiplication and factorising quadratic expressions. Complementary methods for factorising expressions that can be done in parallel are also explored. Unit 15 engages the students in analysing the solutions of some multiplication questions in order to identify any relationship between the numbers in the factors and the numbers in the product. In Unit 16 students will see that dividing algebraic expressions is a skill that can be used when the area of a rectangle and the length of one side are known and the length of the second side is required. Unit 17 introduces the idea that when the product of two numbers is zero then one (or both) of the numbers must be zero. When this is linked to the earlier work on factorising quadratic expressions the students access the skills of solving quadratic equations where the quadratic can be resolved into its factors. The students solve one quadratic equation in Unit 18 and encounter many more in Unit 19. In Unit 20 the need for a method other than factorisation is shown and the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, is introduced as one means of solving all quadratic equations. Transformations of quadratic functions and the information that can be gleaned when quadratic functions are presented in different forms is explored in Unit 21. This idea is continued in Unit 22 where the difference of two squares is explored from a number of viewpoints.

Cubic Expressions, Exponential Relationships and some key skills in Algebra
Cubic expressions are dealt with in Unit 23 and exponential relationships appear in Unit 24. Any remaining work that needs to be done in rearranging formulae, algebraic fractions or factorising by grouping is contained in Units 26, 27 and 28.

Contents

General Overview

The central idea in this document is that it might be better for students to understand why a skill is required before learning it. In addition to endeavouring to make functions the focus of the document, multiple representations are used throughout and some links back to what can be done when students are studying Number to help students understand algebraic concepts are made. Therefore, the learning outcomes of both Strands 4 and 5 are included in this document. The intention is that by the time students are "finished" their work on Algebra that they are also "finished" their work on Functions. The remainder of this overview looks at (a) the flow of some Units in this document, (b) the rationale for the positioning of some algebraic skills (c) an outline of the Units and (d) key features of functions.

(a) The Flow of Some Units in this Document

Broadly speaking, the flow below is used in many Units in this document to show students the need for the algebraic skills they are about to learn prior to learning the skill. Throughout the document multiple solution strategies are used. In many cases students can find the answer to various questions by analysing a table or by interpreting a graph before engaging with the problem using an algebraic method. This means that the algebraic method should make sense to the students as they already know what the answer should be. In many cases too, the need for an algebraic method is obvious as other methods become be too tedious when the numbers are large or prove inaccurate when the answer is not a whole number. Students can bring their own thinking to many of the problems before they are introduced to the formal algebraic approach. By doing so, the students get a greater sense of what they are doing and why they are doing it, can recognise the value of thoughtful engagement with problem solving and appreciate that algebraic techniques offer them incredibly powerful ways of tackling problems. Students should then have a greater appreciation of where algebra fits into mathematics as a whole and how understanding algebraic relationships and techniques is worthwhile.

1. Students engage with a problem

Students engage with a problem through (i) whole class discussion led by teacher questioning, (ii) working in groups, (iii) individual work or (iv) a combination of some or all of the above. Following this, the students implement strategies such as drawing diagrams, analysing a table, trial and improvement or interpreting a graph. The effectiveness of these strategies are then compared by the students through discussion.

2. Students see the need for a new strategy

The students engage with a follow-on problem where the limitations of earlier strategies become apparent and the need for a new strategy is obvious.

3. Students are guided by the teacher to learn the new strategy

Students are guided by the teacher to learn the new strategy. In this document the new strategy will always incorporate algebraic solutions.

4. Students compare the new strategy with the previous strategies

Students compare the new strategy with the previous ones to see the advantages of the algebraic approach and to recognise that algebraic solutions can be checked by using other methods.

(b) Rationale for the position of some algebraic skills

It is acknowledged that students may have prior knowledge (of using the laws of indices when multiplying numbers with a common base or using the distributive law when working in number, for example), but it is envisaged that moving from expanding expressions of the type $3(x + 2)$ to more complex examples like $x(x + 2)$, $(x + 2)(x + 3)$ or $(x + 2)(x^2 + 3x + 4)$ is delayed until the need for such expansions is obvious. For this reason, work on linear functions is treated separately from that on quadratic and cubic functions. This also brings the additional benefit that when students encounter $(x + 2)(x + 3)$ for the first time they will engage with it from a number of perspectives rather than merely as an entity demanding the application of some algebraic skill.

Once a skill is learned in one Unit it can then be used in all subsequent Units. For example, substitution is learned in the "Dots Activity" in Unit 1 of *Algebra Through the Lens of Functions* Part 1 of 2 and while it might not be explicit in the Units that follow it should be seen as a key skill in developing understanding in those Units too.

(c) Key Features of Functions

Throughout the document the key features of functions are referred to. These features can be used when analysing functions so students can use the same criteria for analysing the different functions they encounter as they progress from first year through to sixth year.
The key features of functions are:
1. The domain and range
2. Where the graph of the function meets the axes.
3. Things that remain constant and those that vary in the function,
4. The behaviour of the graph of the function
5. The rate of change of the function

Note: Average rates of change can be used to begin the discussion about how rates of change can be positive, negative or zero and how this can be used to decide if the function is increasing, decreasing or neither. This work can be continued at senior cycle when the slope of the tangent to the function is dealt with.

How to use this document

Throughout the document there are hyperlinks to useful resources, for example, booklets of visual patterns, matching activities, Teacher Resource Booklets from the various Project Maths workshops and Teaching and Learning Plans. Clicking on a hyperlink will bring you to the resource.
The document also contains boxes entitled "Number Work" which provide ideas that should be used when students are studying number, per se, but that can also help students see properties of number that are important for algebra. They are included in this document so that they can be revisited when students encounter the related concept in algebra.
Each Unit contains a number of Sample Problems that can be used with students.

Unit 11: A Quadratic Problem – Forming a Quadratic Expression and Equation and recognising that additional skills are required to Solve the Equation

In this Unit students will:

- use visual patterns to construct a quadratic relationship
- use a table to solve quadratic equations
- use a graph to solve quadratic equations
- use trial and improvement to solve quadratic equations
- appreciate that they may not yet have the algebraic skills to solve quadratic equations algebraically

There are two main goals for this Unit:

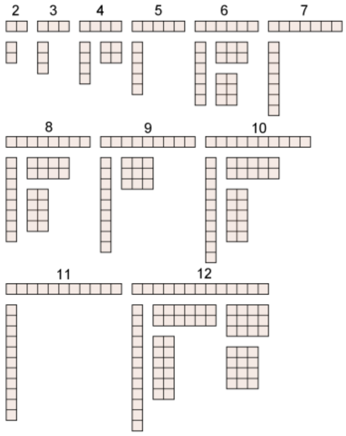
1. To enable students to use visual patterns to construct quadratic relationships.
2. To motivate the need for sufficient algebraic skills to solve quadratic equations.

Note: At this stage students will have done no work on x^2 prior to this, other than when doing indices and looking at how to write numbers as a product of its factors.

Number Work

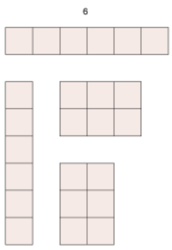
Understanding factors, prime, composite and square numbers will help students with this activity and ones that occur later; for example, simple factorising, factorising by grouping and factorising quadratics.

By building all the possible rectangles from whole numbers of unit squares (for example, 2, 3, 4, 5, 6, 7, etc.) students will see that some numbers only have a very limited choice in how a rectangle can be built, for example, 2, 3, 5, 7 etc. (*prime numbers*).



Some numbers have more choice in how a rectangle can be built, for example the set of composite numbers {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, . . .}.

A subset of the set of composite numbers have the property that they can be arranged into a square. This is the set of square numbers {4, 9, 16, 25, . . .}. (*square numbers*).



A discussion can be had about how the number 6 can be represented as 3 groups of 2 and also as 2 groups of 3 to illustrate the commutative law.

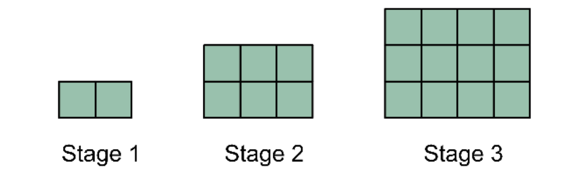
For many questions involving visual patterns a variety of equivalent expressions can be found for the pattern. Colour is used to give hints as to various ways of seeing the pattern.
"Next, Near, Far, Any" (see Unit 1 of *Algebra Through the Lens of Functions* Part 1 of 2) can be used to help students to form the relationship between the stage number and the number of squares. Once even one expression has been formed, substitution can be practiced, for example, how many squares are in the 10th stage?

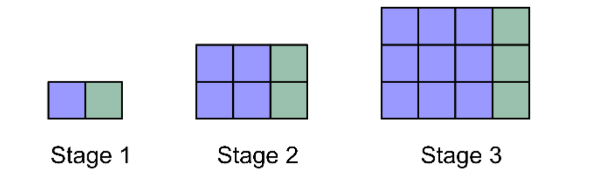
Notes:

1. Four different sample problems are shown below. The minimum students should deal with here is Sample Problem 1 and Sample Problem 2. Sample Problem 2 will be analysed in detail in this Unit. It is advised that Sample Problems 3 and 4 be analysed as it will help to deepen students' understanding of how algebraic expressions are constructed. The analysis might be deferred until the students have learnt some additional skills and concepts, including solving quadratic equations.
2. A Student Workbook: *Visualising Patterns for Quadratic Relationships* is available here. Larger versions of the images are available in PDF form here and as an interactive PowerPoint here.
3. If students haven't yet covered quadratic expressions, they will need help to move from expressing $x(x)$ to its equivalent form of x^2.

Sample Problem 1

Describe the relationship between the stage number and the number of squares



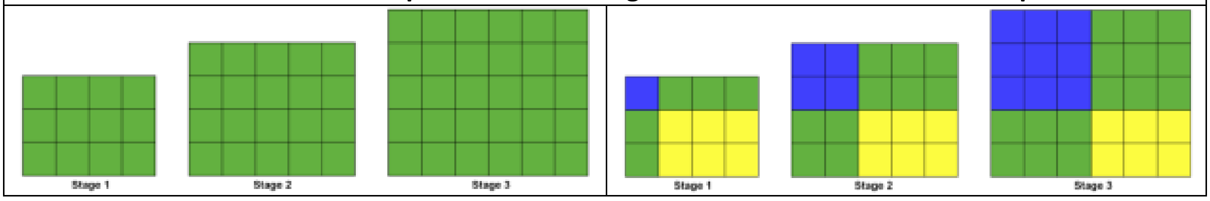


$x(x+1)$ can be seen in the first diagram as the height is the stage number and the width is one more than the stage number. Questioning can be used to help students see how to express the height, width, and area of each stage in terms of the stage number.
Questioning can be used to help students. Such questions might include:
What is the relationship between the stage number and the height of each stage?
What is the relationship between the stage number and the width of each stage?

The colour in the second diagram makes it easier to see $x(x+1)$ in its equivalent form of $x^2 + x$
The blue part contains x^2 squares.
The green part contains x squares.
Questioning can be used to help students see each part.
What is the relationship between the stage number and the number of blue squares in each stage?
What is the relationship between the stage number and the number of green squares in each stage?

Sample Problem 2

Describe the relationship between the stage number and the number of squares

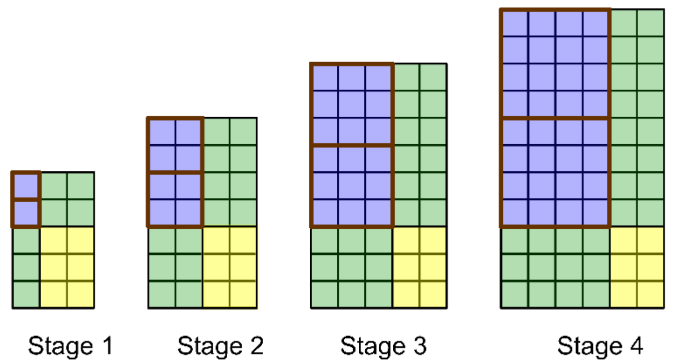


$(x + 2)(x + 3)$ can be seen in the first diagram as the height is always two more than the stage number and the width is always three more than the stage number.
Questioning can be used to help students see each dimension.
What is the relationship between the stage number and the height of the rectangle in each stage?
What is the relationship between the stage number and the width of the rectangle in each stage?

The colour in the second diagram makes it easier to see the total number of squares is the sum of a variable squared, plus five times a variable and a constant i.e. $x^2 + 5x + 6$.
The blue part has x^2 squares.
The green parts contain $5x$ squares.
The yellow part contains 6 squares.
Questioning can be used to help students see each individual area within each stage. Such questions might include:
How many yellow squares are in each stage?
What is the relationship between the stage number and the number of green squares in each stage?
What is the relationship between the stage number and the number of blue squares in each stage?

Sample Problem 3

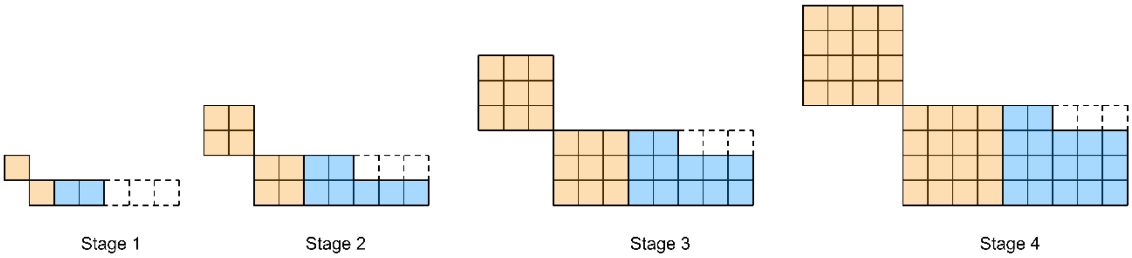
Describe the relationship between the stage number and the number of squares



In the following commentary the variable x refers to the stage number and:
The colour in the diagram makes it easier to see that the area of the rectangles can be expressed as $2x^2 + 7x + 6$.
The area in blue, in each stage has an area that can be expressed as $2x^2$.
The green parts always contains $7x$ squares.
The yellow part always contains 6 squares.
Questioning can be used to help students recognise the link between the stage number and the area of each coloured part. Such questions might include:
How many yellow squares are contained in each stage?
What is the relationship between the stage number and the number of green squares in each stage?
What is the relationship between the stage number and the number of blue squares in each stage?

Sample Problem 4

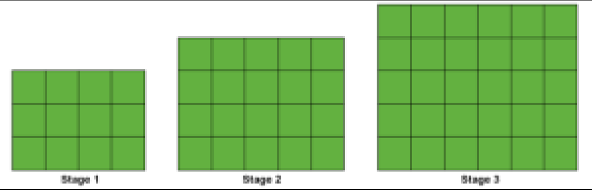
Describe the relationship between the stage number and the number of shaded squares

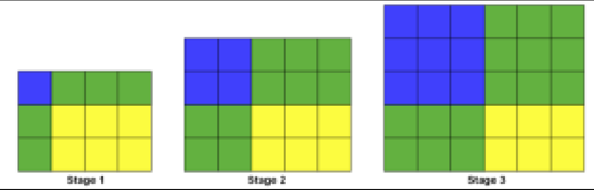


In the following commentary the variable x refers to the stage number and:
The colour in the diagram makes it easier to see that the number of shaded squares in each stage can be expressed as $2x^2 + 5x - 3$.
The number of orange squares in each stage can be expressed as $2x^2$.
The remaining section of each stage (including the cutaway part) contains $5x$ squares.
The cutaway part has 3 squares.
The total number of shaded squares in each stage is $2x^2 + 5x - 3$.
Questioning can be used to help students recognise the link between the stage number and the number of squares in of each coloured part. Such questions might include:
What is the relationship between the stage number and the number of orange squares in each stage?
What is the relationship between the stage number and the number of blue or white squares in each stage?
How many white squares are contained in each stage? How many squares are cutaway in each stage?

The remaining part of this Unit illustrates the need to incorporate traditional algebraic skills in solving quadratic equations.

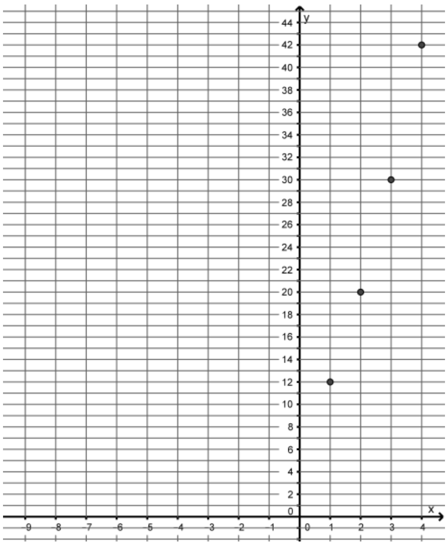
Describe the relationship between the stage number and the number of squares





After the students form expressions they can be given a [partially filled in table]. Students could construct the table of points themselves but this partially filled in table will be central to some later Units so it is seen as worthwhile to scaffold their work here. There is no need for students to fill in the blank rows near the top of the table at this stage as this will be dealt with in Unit 18.
The students should then graph the points (1, 12), (2, 20), (3, 30) and (4, 42) on squared paper where the x-axis of the graph should go from –9 to 4 and the y-axis should go from –5 to 45. Other points will be plotted on this graph in Unit 18.

Stage Number x	$(x+2)(x+3)$	**Number of Squares**	**Rate of Change of the Outputs**	**Change of the Change of the Outputs**
1	3(4)	12	+8	
2	4(5)	20	+10	
3	5(6)	30	+12	
4	6(7)	42		



The table can be used to show that the relationship is not linear. A straight-edge can be used to show that the points plotted on the graph are not collinear.
We will look at the change of the change in Unit 19.
*****The next two questions will be used as motivation for the content of next few Units:*****
(i) Which stage has 42 squares?
(ii) Which stage has 156 squares?

Question (i) could be answered using many methods. Four are outlined below:

- Sketching possible rectangles where one side is one unit longer than the other and seeing which one has 42 squares
- Trial and improvement using either $(x+2)(x+3)$ or x^2+5x+6
- Reading the table to find which stage number has an output of 42 squares
- Interpreting the graph to see what input is required for an output of 42.

Question (ii) can be solved using similar methods to question (i). Sketching the rectangles would be inefficient. Trial and improvement could be a better strategy. Continuing the table would take a long time. Extending a graph to have an output of 156 would be difficult.

Analysing the available strategies for question (ii) could lead students to see that maybe there is a need for a different strategy. It could be said to students that using algebra formally could help with solving this problem and also that they will have to learn a few skills before they are able utilise algebraic skills to solve these problems.

Unit 12: Quadratic Expressions in Multiple Representations

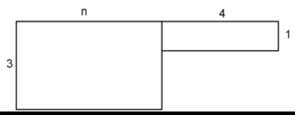
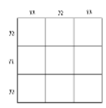
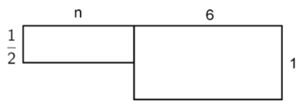
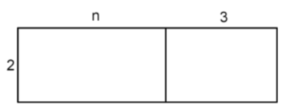
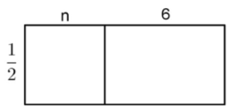
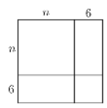
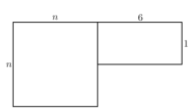
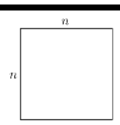
In this Unit students will:

- make connections between any two representations (Words, Symbols, Area) of a given expression (Linear or Quadratic)
- draw an area representation when given a quadratic expression expressed in words or symbols
- visualise like and unlike terms
- visualise the distributive law
- use letters to represent quantities that are variable
- visualise some transformational activities e.g. collecting like terms and expanding
- understand if two algebraic expressions are equivalent or not

There are two central activities to this Unit: **A.** Matching and **B.** Drawing. The activities aim to help students visualise the ***distributive law for algebraic expressions***, like and unlike terms, and equivalent expressions.

A. Matching

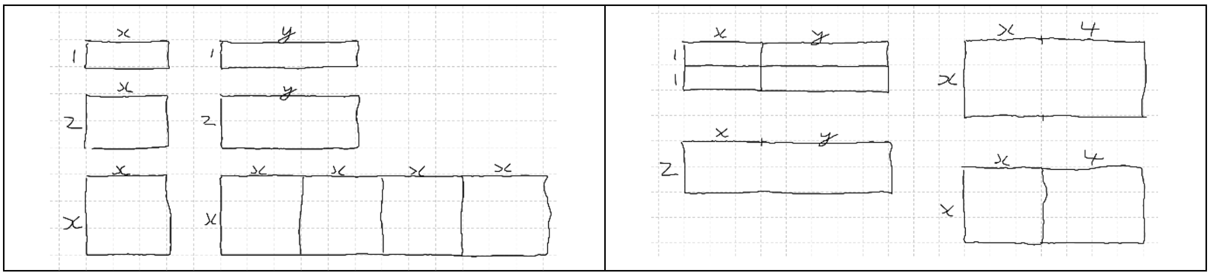
This extends the previous matching activity by including quadratic expressions. This activity focuses on just three of the representations (Words, Symbols and Area). The activity could be used again at a later stage with a Table and Graph. In order to make this activity challenging for more able students, solutions that include fractions are also included. The matching activity below can be found here.

A1 	**E6** $3n + 4$	**W3** Multiply n by three, then add 4.
A2 	**E5** $9n^2$	**W2** Multiply n by 3, then square the answer.
	E11 $(3n)^2$	
A3 	**E10** $\frac{n}{2} + 6$	**W8** Divide n by two, then add 6.
A4 	**E1** $2(n + 3)$	**W4** Multiply n by two, then add 6.
	E8 $2n + 6$	**W9** Add three to n, then multiply by two.
A5 	**E2** $\frac{n+6}{2}$	**W7** Add six to n, then divide by two.
		W10 Add six to n, then multiply by a half.
A6 	**E4** $n^2 + 12n + 36$	**W6** Add six to n, then square the answer.
	E9 $(n + 6)^2$	
A7 	**E7** $n^2 + 6$	**W1** Square n, then add six.
A8 	**E3** n^2	**W5** Square n.

B. Drawing

A similar activity to this was used in Workshop 5.

Draw the following arrays: $x, y, 2x, 2y, x^2, 4x^2, 2x + 2y, 2(x + y), x(x + 4), x^2 + 4x$ where $x \neq y$.



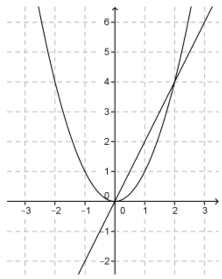
Notes:

1. Using squared paper makes this activity much easier.
2. Prompting students to see x as $1x$ or $+1x$ and y as $1y$ or $+1y$ will help some students with this activity.
3. $4x^2$ could be represented as a row of four squares each with side length x or one square with a side length $2x$ i.e. $(2x)^2 = 4x^2$.
4. x^2 will take the shape of a square. $2x$ will take the shape of an oblong rectangle unless $x = 2$.
5. An interesting discussion point will arise if some students draw x as two boxes across; the area of x^2 will be equal to the area of $2x$. $x^2 = 2x$ when $x = 0$ or $x = 2$. For all other values of x it should be clear to students that $x^2 \neq 2x$.

A table can be used to demonstrate that $x^2 = 2x$ for only two values of x.

x	$2x$	x^2
0	0	0
1	2	1
2	4	4
3	6	9
4	8	16
5	10	25

If it was thought that students could understand a quadratic graph then graphs of $f(x) = 2x$ and $g(x) = x^2$ could be drawn. The functions will have common values when $x = 0$ and when $x = 2$.



Students will not know at this stage how to solve for x in the equation $x^2 = 2x$.

Unit 13: Simplifying the Sum and Difference of Like Terms and Simple Factorising using the Areas of Rectangular Shapes

In this Unit students will:

- visualise the sum the sum and difference of like terms
- simplify the sum and difference of like terms
- simplify expressions of the form:
    - $(ax + by + c) \pm \cdots \pm (dx + ey + f)$
    - $(ax^2 + bx + c) \pm \cdots \pm (dx^2 + ex + f)$
    - $a(bx + cy + d) + \cdots + e(fx + gy + h)$
    - $a(bx^2 + cx + d)$

This continues the work of the previous Unit where students look at how the dimensions and areas of rectangular shapes can be expressed, for example, $x(x + 4) = x^2 + 4x$, $x(x)$, $x(2x)$, $x(2x + 1)$ etc.

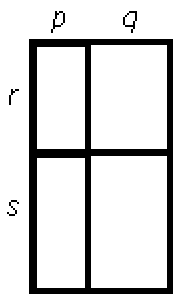
While the areas of shapes can be added and subtracted and composed and decomposed, it is recommended that when simplifying the expressions bulleted immediately above, an algebraic approach rather than one involving the area model is used.

Unit 14: Multiplying Expressions and Factorising by Grouping to Express the Areas of Rectangular Shapes Concisely

In this Unit students will:

- multiply expressions to form quadratic expressions
- appreciate the efficiency of expressing the area of rectangles in factored form
- factorise by grouping to express the area of rectangles concisely

This Unit is based on the some matching activities from the Workshop 5 Teacher Resource Booklet and includes work with area diagrams that contain a number of variables. Expressing the areas in equivalent forms should engage students in factorising expressions.



The area of the diagram opposite is $pr + qr + ps + qs$ (adding the four parts). It can also be expressed as $(p + q)(r + s)$ (length by breadth). $r(p + q) + s(p + q)$ (top plus bottom) and $p(r + s) + q(r + s)$ (left plus right) are other possible equivalent expressions.

Note: Students have already seen how to write $6x + 3$ in equivalent form as $3(2x + 1)$ with the aid of a diagram. The use of diagrams may also assist students in recognising $r(p + q) + s(p + q)$ as an intermediate step in factorising by grouping.

Since the diagram above is in the shape of a rectangle it should be possible to express its area as the product two factors i.e. length by breadth. Some students will be able to see the answers in factored form very easily. However, all students need to experience the process of rewriting $pr + qr + ps + qs$ as $(p + q)(r + s)$. Factorising results in the area of a rectangle being expressed concisely as a product of its dimensions. A diagram containing three or four terms could help illustrate this more clearly: The expression $(a + b + c)(p + q + r + s)$ is more concise than an equivalent expression containing the sum of twelve terms. Using the factored form makes it easier to evaluate areas, as less substitution is required. For example, it is more efficient to evaluate $(a + b + c)(p + q + r + s)$ than $ap + aq + ar + as + bp + bq + br + bs + cp + cq + cr + cs$ when the values for a, b, c, p, q, r and s are known.

Steps for Multiplying $(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{r}+\boldsymbol{s})$ using the Array Model

Step 1

	r	$+s$
p		
$+q$		

Step 2

	r	$+s$
p	pr	
$+q$		

Step 3

	r	$+s$
p	pr	$+ps$
$+q$		

Steps 4 and 5

	r	$+s$
p	pr	$+ps$
$+q$	$+qr$	$+qs$

Thus $(p+q)(r+s) = pr + ps + qr + qs$

Multiplying $(p+q)(r+s)$ by another, more traditional method, for example, splitting the first bracket, should also be shown in parallel with the array model method.

Steps for Factorising $\boldsymbol{pr}+\boldsymbol{ps}+\boldsymbol{qr}+\boldsymbol{qs}$ using the Array Model

Step 1

pr	$+ps$
$+qr$	$+qs$

Step 2

p	pr	$+ps$
	$+qr$	$+qs$

Step 3

	r	$+s$
p	pr	$+ps$
	$+qr$	$+qs$

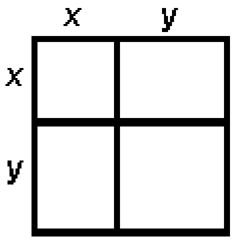
Step 4

	r	$+s$
p	pr	$+ps$
q	$+qr$	$+qs$

Thus $pr + ps + qr + qs = (p+q)(r+s)$

Factorising $pr + ps + qr + qs$ by another, more traditional method, should also be shown in parallel with the array model method.

Expressions of the form $(x+3)^2$, $x^2 + 3x + 3x + 9$, $(x+y)^2$ and $x^2 + xy + xy + y^2$ could also be investigated.

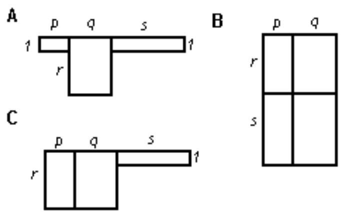


Another matching exercise with questions similar to the one below could also be used to see equivalent and non-equivalent forms. This question is from the [Workshop 5 Teacher Resource Booklet].

(iii) $(p+q)\times(r+s)$ ☐

$p+q\times r+s$ ☐

$(p+q)\times r+s$ ☐



Unit 15: Multiplying Two Linear Expressions and Factorising Quadratic Expressions

In this Unit students will:

- Expand expressions of the form
  - $(x+2)(x+3)$
  - $(x-4)(x+9)$
  - $(x-5)(x-7)$
- Factorise expressions of the form
  - x^2+5x+6
  - $x^2+5x-36$
  - $x^2-12x+35$

Note: There is a small number of algebraic skills that need to be made explicit here before showing students why they need the skill. All students should be taught to expand expressions of the form $(x+2)(x+3)$ and $(x-4)(x+9)$ and factorise quadratic expressions like x^2+5x+6 and $x^2+5x-36$. All other multiplciation and factorising can be left until solving quadratic equations is encountered. This will create an awareness of why the skill of factorising the quadratic is required. In motivating the need for these skills, it could be pointed out to students that the skill of factorising is valuable when solving quadratic equations and simplifying expressions of the form $\frac{x^2+5x+6}{x+2}$, for example.

Ultimately, the goal of this Unit is to link expanding products of the form $(x+2)(x+3)$, $(x-4)(x+9)$, and $(x-5)(x-7)$ with factorising expressions like x^2+5x+6, $x^2+5x-36$ and $x^2-12x+35$.

Note: The list of expansions directly above should be regarded as the minimum that students should encounter at this stage. However, it is advised that not too much time be spent expanding further expressions of these types at this stage as they will be encountered in greater detail later.

Multiplying the expressions can be approached in many ways. Two complementary methods, which can be done in parallel, are shown below.

$(x+2)(x+3)$

= $x(x+3)+2(x+3)$

= $x^2+3x+2x+6$

= x^2+5x+6

	x	$+3$
x	x^2	$+3x$
$+2$	$+2x$	$+6$

Number Work

12×13 can be expressed as (10+2)(10+3) and can also be drawn as a rectangle with dimensions (10+2) and (10+3).

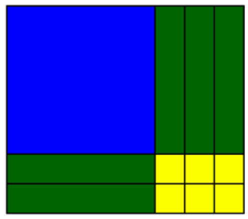
12×13
= (10+2)(10+3)
= 10(10+3)+2(10+3)
= 100+30+20+6
= 156

Any work done previously with factors, multiples, prime numbers, composite numbers and square numbers would also help when students are learning how to factorise quadratic expressions.

Below are two options for factorising the expressions of the form $x^2 + 5x - 36$ and $x^2 - 12x + 35$.

<u>Option 1: With Algebra Tiles:</u>

$x^2 + 5x + 6$. Ask students to arrange one x^2 tile, five x tiles and six unit tiles into a rectangle. How many ways can it be done? What are the dimensions of the rectangle?



Note: It is also possible to use Algebra Tiles to visualise factorising expressions like $x^2 + 5x - 36$ and $x^2 - 12x + 35$.

<u>Option 2: Without Algebra Tiles:</u>

Look at the solutions of a few questions like the ones below and ask students do they spot any relationship between the numbers in the factors and the numbers in the product? Then hide the questions below and ask them to factorise an expression (it could even be one of the expressions from below).

With this option you could still ask students to draw an x^2, five xs and six units in the shape of a rectangle i.e. make a diagram that is similar to Algebra Tiles.

$(x+2)(x+3)$	$(x+6)(x+7)$	$(x-4)(x+9)$	$(x-5)(x-7)$
$= x(x+3) + 2(x+3)$	$= x(x+7) + 6(x+7)$	$= x(x+9) - 4(x+9)$	$= x(x-7) - 5(x-7)$
$= x^2 + 3x + 2x + 6$	$= x^2 + 7x + 6x + 42$	$= x^2 + 9x - 4x - 36$	$= x^2 - 7x - 5x + 35$
$= x^2 + 5x + 6$	$= x^2 + 13x + 42$	$= x^2 + 5x - 36$	$= x^2 - 12x + 35$

Notes:

1. The array model for factorising is a visual representation of the guide number method of factorising.

Factorise: $x^2 - 12x + 35$

$= \quad x^2 - 12x + 35$

$= \; x^2 - 5x - 7x + 35$

$= x(x-5) - 7(x-5)$

$(x-7)(x-5)$

Rough Work

Guide Number = $(1)(35) = 35$

$-1 \times -35 = 35 \quad -1 + (-35) = -36$

$-5 \times -7 = 35 \quad -5 + (-7) = -12$

Step 1

x^2	
	$+35$

Step 2

x^2	$-5x$
$-7x$	$+35$

Step 3

	x	-5
x	x^2	$-5x$
	$-7x$	$+6$

Step 4

	x	-5
x	x^2	$-5x$
-7	$-7x$	$+35$

$(x-7)(x-5)$

Unit 16: Dividing Algebraic Expressions

In this Unit students will:

- appreciate how the division of algebraic expressions arises from the need to solve problems relating to the area of rectangles
- divide one dimension of a rectangle into the expression for its area
- carry out operations of the form:
    - $(x^2 + bx + c) \div (dx + e)$
    - $(ax^2 + bx + c) \div (dx + e)$

Dividing algebraic expressions can be introduced to students as arising from the need to solve problems in area. If the area of a rectangle is known to be $x^2 + 5x + 6$ and one of its dimensions is $x + 2$ the other dimension can be found using division. One approach to finding the missing dimension is to use the area/array model:

Step 1

x	x^2	
$+2$		$+6$

Step 2

	x	
x	x^2	
$+2$		$+6$

Step 3

	x	$+3$
x	x^2	
$+2$		$+6$

Additional Step

	x	$+3$
x	x^2	$+3x$
$+2$	$+2x$	$+6$

Note:

1. Despite the occurrence of terms with negative coefficients, the array method offers a good means of tackling these problems.
2. It should be emphasised that simplifying a quadratic expression divided by linear expression can be done by factorising the quadratic expression as well, and ***dividing above and below by the common factor*** to get an equivalent expression.

Unit 17: A Concept Required for Solving Quadratics – A Special Property of Zero

In this Unit students will:

- see that when the product of two numbers is zero then one or both of the numbers must be zero

The following problem could be posed to students: "We need to use some skills we have learnt and learn one new concept to solve the problems $(\boldsymbol{x}+\mathbf{2})(\boldsymbol{x}+\mathbf{3})=\mathbf{42}$ and $(\boldsymbol{x}+\mathbf{2})(\boldsymbol{x}+\mathbf{3})=\mathbf{756}$ using algebra. Part of your homework tonight is a challenge. Your challenge is to find two numbers, neither of which is zero, whose product is zero. Be prepared to defend your reasoning and to discuss the challenge tomorrow in class."

The purpose of this challenge is for students to appreciate the special property of zero. Hopefully this challenge will be memorable as it will be frequently referred to hereafter. Once students understand that when the product of two numbers is zero then one or both of the numbers must zero they have the acquired a key skill required to solve quadratic equations.

Alternatively, a "game" could be used where the teacher says he/she can predict the product of any number the students' choose multiplied by a number the teacher chooses. The teacher can write the product on a piece of paper. Then he/she can ask what number the student chose. Then the teacher can show the product he/she has written on the piece of paper. The teacher chooses zero. This can be repeated until the students understand the concept.

Note: Investigating a number of examples is not a proof. See "From Discovery to Proof" on page 79 of the Junior Certificate Syllabus.

A proof that the product of two numbers, neither of which is zero, is never zero is shown below:
To show if $xy=0$ then $x=0$ or $y=0$.
Suppose $xy=0$ for $x\neq 0$, $y\neq 0$
$xy=0$
$\frac{xy}{y}=\frac{0}{y}$ Dividing both sides by y. **Note:** $y\neq 0$
$x=0$
This contradicts the given fact that $x\neq 0$.
A similar contradiction would arise if both sides were divided by x.
Thus if $xy=0$ then $x=0$ or $y=0$, as required.
Note: The proof above is a proof by contradiction, which is only specifically mentioned in the Leaving Certificate Higher Level syllabus.

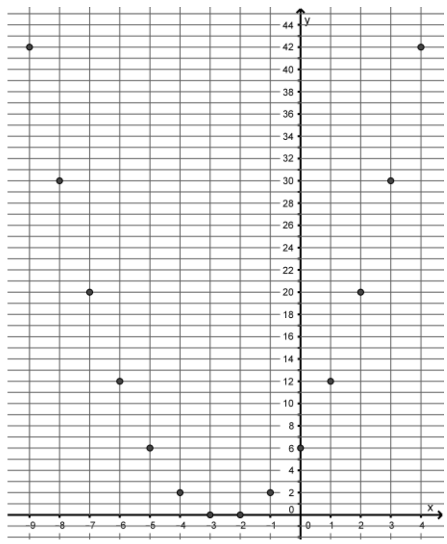
Unit 18: Solving a Quadratic Equation Algebraically

In this Unit students will:

- solve quadratic equations algebraically
- see how the algebraic solution relates to solutions found through other methods

The students should be reminded at this stage that they have acquired the skills required to solve quadratic equations algebraically and are reintroduced to the problems:
(i) Which stage has 42 squares?
(ii) Which stage has 156 squares?
The use of dynamic software is immensely helpful here. Solving $x^2 + 5x + 6 = 42$ is the same as comparing the functions $f(x) = x^2 + 5x + 6$ and $g(x) = 42$. The common value(s) of both functions will occur for the same values of x. These same values of x are also where $h(x) = x^2 + 5x - 36$ and $i(x) = 0$ are equal.
Note: Students have not transformed quadratic functions yet but they have seen the effect of changing c in the linear function $y = mx + c$.
Utilising the special property of zero means that $x^2 + 5x + 36 = 42$ can be solved.
$x^2 + 5x + 6 = 42$ is equivalent to $x^2 + 5x - 36 = 0$ by subtracting 42 from both sides, and now $(x - 4)(x + 9) = 0$, by factoring the quadratic.
Finally, since the product of the factors is zero $x - 4 = 0$ or $x + 9 = 0$ and $x = 4$ or $x = -9$ are the roots/solutions of the equation $x^2 + 5x - 36 = 0$ and, consequently, of the equation $x^2 + 5x + 36 = 42$.
In the context of the question only $x = 4$ makes sense. Disregard $x = -9$. Pattern 4 has 42 squares.
To illustrate where the $x = -9$ comes from, the partially filled in table and graph from Unit 11 can be used. The points (1, 12), (2, 20), (3, 30) and (4, 42) were completed earlier in the table and graph. They should now go back and extend this table back to (–9, 42) and plot these points.

Stage Number x	$(x+2)(x+3)$	Number of Squares	Rate of Change of the Outputs	Change of the Change of the Outputs
–9	–7(–6)	42		
–8	–6(–5)	30	–12	
–7	–5(–4)	20	–10	
–6	–4(–3)	12	–8	
–5	–3(–2)	6	–6	
–4	–2(–1)	2	–4	
–3	–1(0)	0	–2	
–2	0(1)	0	0	
–1	1(2)	2	+2	
0	2(3)	6	+4	
1	3(4)	12	+6	
2	4(5)	20	+8	
3	5(6)	30	+10	
4	6(7)	42	+12	



The language of Equation→ Factors → Roots can be emphasised here in preparation for working in reverse at JCHL. Topic 4.7 has the learning outcome “form quadratic equations given whole number roots”.
$x^2 + 5x + 6 = 156$ can be looked at in a similar way to emphasise that algebra is efficient (it would be tedious to continue the table to when $x = 10$, or use trial and error to get $x = 10$, or estimate from a graph that would take quite a while to draw).

Notes:

1. The table, graph and substitution methods can be used to check the answer we get from the most efficient and accurate method i.e. solving algebraically.
2. To really sell the idea that the algebraic method is best, a continuous function could be used i.e. one that uses a domain of R or R^+ and where the solutions are not necessarily integers.
3. $x^2 + 5x - 36 = 0$ and $x^2 - 12x + 35 = 0$ could be kept in mind as questions to do as students have factorised both quadratic expressions before.

Unit 19: Factorising More Quadratic Expressions and Solving More Quadratic Equations

In this Unit students will:

- factorise expressions such as $ax^2 + bx + c$, $a \in \mathbb{N}, b, c \in \mathbb{Z}$
- solve quadratic equations

This would be a good time to factorise expressions such as $ax^2 + bx + c$, $a \in \mathbb{N}, b, c \in \mathbb{Z}$

Additional visual patterns questions can be found in the Student Workbook: *Visualising Patterns for Quadratic Relationships* and in this document Patterns: A Relations Approach to Algebra (which includes growing squares (x^2), growing rectangles $x(x + 1)$, Staircase towers (triangular numbers) $\frac{1}{2}x(x+1)$ and others).
Real world context questions can be found in Modular Course 3. The following question can also be looked at using multiple representation: “Farmer Giles has 14 m of wire mesh fencing. He wants to enclose a herb garden in a rectangular shape. What is the area of the largest possible garden he can create?”.

Some of the skills work to improve procedural fluency for quadratic equations that can be simplified to $ax^2 + bx + c = 0$ could be done at this stage.

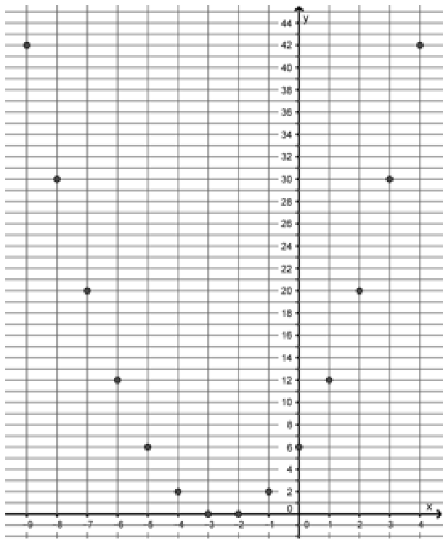
Extending Students' Appreciation of Quadratic Equations and Functions

The multifaceted approach adopted to solving quadratic equations creates opportunities to explore and exploit links to a number of related areas. Below are three areas that can, and should, be investigated in parallel with work on improving procedural fluency for solving quadratic equations. These three areas are:

(i) Key Features of Quadratic Functions
(ii) Graphical Solutions to Algebraic Inequalities
(iii) Analysing the Table of Values of Quadratic Functions in Greater Depth

(i) Key Features of Quadratic Functions:

Stage Number x	$(x+2)(x+3)$	Number of Squares	Rate of Change of the Outputs	Change of the Change of the Outputs
−9	−7(−6)	42		
−8	−6(−5)	30	−12	
−7	−5(−4)	20	−10	
−6	−4(−3)	12	−8	
−5	−3(−2)	6	−6	
−4	−2(−1)	2	−4	
−3	−1(0)	0	−2	
−2	0(1)	0	0	
−1	1(2)	2	+2	
0	2(3)	6	+4	
1	3(4)	12	+6	
2	4(5)	20	+8	
3	5(6)	30	+10	
4	6(7)	42	+12	



Domain: What is the domain of the function?
Range: What is the range of the function?
Where the graph of the function meets the axes: Where does the function intersect the x and y axes?
Constant(s): What is constant in the function?
Variable(s): What is varying in the function?
Behaviour of the Graph of the Function: For what values of x is does the function have outputs that are (i) Positive, (ii) Negative or (iii) Zero?
For what values of x is the function (i) increasing or (ii) decreasing?
Does the function have a turning point?
How would you describe the shape of the graph of the function?
Turning Point
Axis of symmetry

The Rate of Change of the Function:
The slope of the tangent to the function is Leaving Certificate material. However, if students are asked the question above "For what values of x is the function (i) increasing or (ii) decreasing?" students should be able to see a link between the fact that the graph's values are decreasing between $x = -9$ and $x = -2.5$ and the "changes" in the outputs of quadratic function in a table are mostly negative in this part of the table i.e. –12, –10, –8, –6, –4, –2 and 0.
These are average rates of change; the change over an interval.

Note: If the function had a domain of $\mathbb{R}$ this function would be continuous and at Leaving Certificate the students could get the instantaneous rate of change at any point.
To focus this on what can be done with Junior Certificate Higher Level students in preparation for Leaving Certificate it would be opportune to ask students “For what values of x is the function (i) increasing or (ii) decreasing?” and see if they can see a link between the graph and the table.

(ii) Graphical Solutions to Algebraic Inequalities
The following learning outcomes from topic 5.2:
“use graphical methods to find approximate solutions where $f(x) = g(x)$ and interpret the results
“find maximum and minimum values of quadratic functions from a graph”
“interpret inequalities of the form $f(x) \leq g(x)$ as a comparison of functions of the above form; use graphical methods to find approximate solution sets of such inequalities and interpret the results”

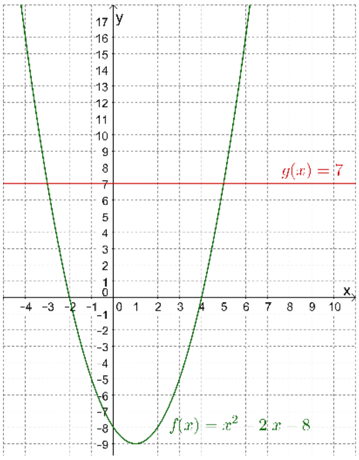
Below are some questions that could be used at this time when students will be making out graphs of quadratic and linear functions to develop understanding of some of the learning outcomes from topic 5.2.

1. Point to where the graphs of the functions intersect.
2. Write down the coordinates of the point of intersection.
3. For what value(s) of x do both functions have the same value?
4. For what value(s) of x do both functions have the same output?
5. Point to the function has its maximum or minimum value.
6. Write down the coordinates of the point on the graph of the function where the function has its lowest (or highest) value.
7. Point to where the quadratic function has higher values than the linear function.
8. Point to where the quadratic function has lower values than the linear function.
9. For what values of x does the quadratic function have the higher values than the linear function?
10. For what values of x does the quadratic function have the lower values than the linear function?
11. For what values of x does the quadratic function have the greater outputs than the linear function?
12. For what values of x does the quadratic function have the greater outputs than the linear function?
(Does the table of values confirm that your answers to the questions above could be correct?)

The questions above could be useful for questions like
“Solve for x in the following equation: $x^2 - 2x - 8 = 7$”

x	$f(x)$
–4	16
–3	7
–2	0
–1	–5
0	–8
1	–9
2	–8
3	–5
4	0
5	7
6	16
2	4(5)
3	5(6)
4	6(7)

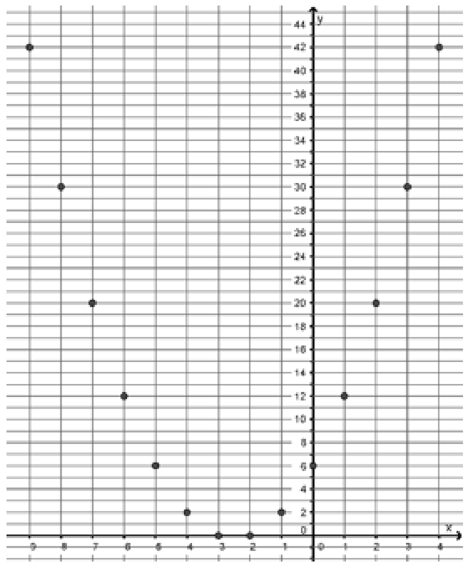
x	$g(x)$
–4	7
–3	7
–2	7
–1	7
0	7
1	7
2	7
3	7
4	7
5	7
6	7
2	7
3	7
4	7



(iii) Analysing the Table of Values of Quadratics in Greater Depth

The table of values for a quadratic could also be analysed in more depth at this stage to do some work on the following learning outcome from topic 4.4: “recognise that a distinguishing feature of quadratic relations is the way the change varies”.
Students could also be encouraged to notice that the change of the change is constant.

Stage Number x	$(x+2)(x+3)$	Number of Squares	Rate of Change of the Outputs	Change of the Change of the Outputs
−9	−7(−6)	42		
−8	−6(−5)	30	−12	
−7	−5(−4)	20	−10	+2
−6	−4(−3)	12	−8	+2
−5	−3(−2)	6	−6	+2
−4	−2(−1)	2	−4	+2
−3	−1(0)	0	−2	+2
−2	0(1)	0	+0	+2
−1	1(2)	2	+2	+2
0	2(3)	6	+4	+2
1	3(4)	12	+6	+2
2	4(5)	20	+8	+2
3	5(6)	30	+10	+2
4	6(7)	42	+12	+2



Unit 20: Solving Quadratic Equations using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In this Unit students will:

- see that not all quadratic equations have integer or rational solutions
- see that the factorisation method for solving quadratic equations will not always work
- solve quadratic equations algebraically using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- check solutions using a table, graph, substitution and the table function on the calculator

Students should be encouraged to look over the solutions they have found to the equations they have solved and see if they notice what type of numbers they are. The answers have all been rational numbers. Students have seen that $x^2 + 5x + 6 = 42$ can be represented by two functions; one quadratic and one linear. A quadratic function and a linear function could be sketched on the board and students could be asked do they think that the solutions will always be rational numbers every time a quadratic function and a linear function intersect?
Students could be asked to solve $x^2 - 5x + 2 = 10$. When they get stuck (because $x^2 - 5x - 8$ cannot be factorised) encourage them to make out a table of values for $f(x) = x^2 - 5x + 2$ to see when the function has an output of 10 and/or make out a table of values for $h(x) = x^2 - 5x - 8$ to see when the function has an output of 0. Graphs could be drawn as well.

As said in the bullet points above, students need to be shown a quadratic equation where factorisation will not work to appreciate that another method is required, for example, $x^2 - 5x - 8 = 0$.

Note: The numbers in the question were chosen carefully to ensure the coefficient of x is negative. This ensures that it will be advantageous to use brackets for all parts of the substitution that is required. If b was 4 then the $-b$ and the b^2 could be written as -4 and 4^2 and some students might then in the future write -5^2 instead of $(-5)^2$. Starting with a negative coefficient of x in the first example means that the $-b$ and the b^2 could be written as $-(-5)$ and $(-5)^2$ which starts good habits for the future.

Unit 21: Transformations of Quadratic Functions and Different Forms of a Quadratic

In this Unit students will:

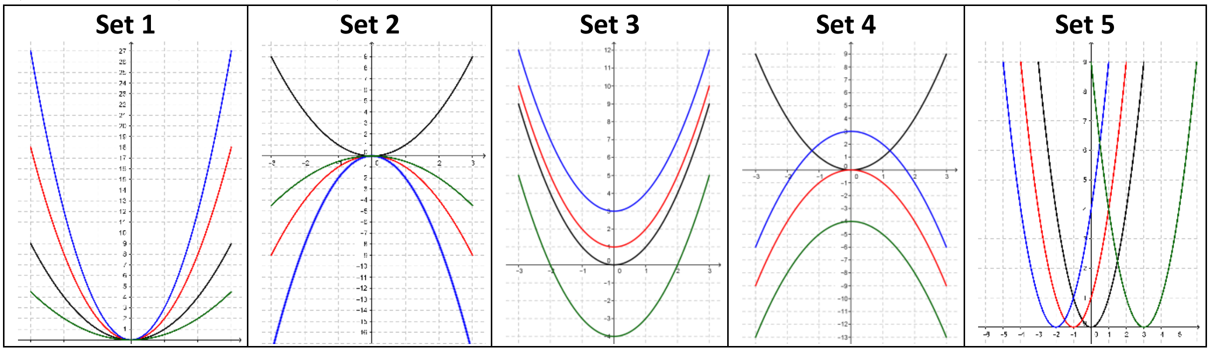
- verbalise the effect of the "a" in $g(x) = ax^2$, using the language of transformation geometry
- verbalise the effect of the "c" in $g(x) = x^2 + c$, using the language of transformation geometry
- verbalise the effect of the "a" in $g(x) = (x + a)^2$, using the language of transformation geometry
- use information of functions of the form $f(x) = ax^2 + bx + c$ or $f(x) = (x + a)(x + b)$ to sketch the function
- form quadratic equations when given whole-number roots

This Unit has three sections (i) Transformations of Quadratic Functions and (ii) Different Forms of a Quadratic Activity 1 (Different Forms of a Quadratic Activity 2 will be in the next Unit) and (iii) Forming Quadratic Equations when Given Whole Number Roots.

(i) Transformations of Quadratic Functions

The sets of functions below are very similar the ones in the Workshop 7 Teacher Resource Booklet. The sets could be divided amongst the class. Each group of students must make out a table for a set of four functions. Each group should draw a set of four functions on the one diagram, using different colours for each function. Each group should then verbalise what $g(x)$, $h(x)$ and $p(x)$ look like when compared to the parent function $f(x) = x^2$. It is not necessary for each student to draw all twenty functions.

Set 1: $f(x) = x^2$, $g(x) = 2x^2$, $h(x) = 3x^2$, $p(x) = 0.5x^2$ all in the domain $-3 \le x \le 3$.
Set 2: $f(x) = x^2$, $g(x) = -x^2$, $h(x) = -2x^2$, $p(x) = -0.5x^2$ all in the domain $-3 \le x \le 3$.
Set 3: $f(x) = x^2$, $g(x) = x^2 + 1$, $h(x) = x^2 + 3$, $p(x) = x^2 - 4$ all in the domain $-3 \le x \le 3$.
Set 4: $f(x) = x^2$, $g(x) = -x^2$, $h(x) = -x^2 + 3$, $p(x) = -x^2 - 4$ all in the domain $-3 \le x \le 3$.
Set 5: $f(x) = x^2$, $g(x) = (x + 1)^2$, $h(x) = (x + 2)^2$, $p(x) = (x - 3)^2$ in the domains $-3 \le x \le 3$, $-4 \le x \le 2$, $-5 \le x \le 1$, $0 \le x \le 6$.



The language of parent function, translate, vertical, horizontal, stretch, compress, shift, scale and reflect can be used:
Example 1: $g(x) = 2x^2$, $-3 \le x \le 3$ could be described as a scaled version of the parent function $f(x) = x^2$, $-3 \le x \le 3$. **Note:** Neither function is "thinner" than the other.
Example 2: $g(x) = -x^2$, $-3 \le x \le 3$ could be described as the image of the parent function $f(x) = x^2$, $-3 \le x \le 3$ under a reflection in the x-axis.
Example 3: $g(x) = (x + 1)^2, -4 \le x \le 2$ could be described as the image of the parent function $f(x) = x^2$, $-3 \le x \le 3$ under a horizontal translation of one unit to the left.

The order of operations is central to these transformations. Throughout all the sets above the pattern 9, 4, 1 0, 1, 4, 9 will be seen repeatedly.

Example 1: The squaring will be performed first in $g(x) = 2x^2$ resulting in 2(9), 2(4), 2(1), 2(0), 2(1), 2(4) and 2(9). Each output of the parent function is then doubled.

Example 2: $h(x) = x^2 + 1$ will result in 9+1, 4+1, 1+1, 0+1, 1+1, 4+1 and 9+1. Each output of the parent function has been increased by one.

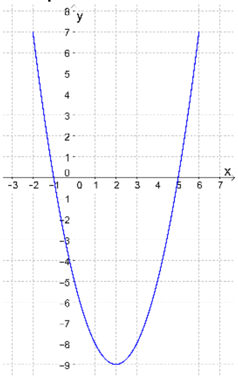
Note: $(x + a)^2$ should be investigated so students can understand that if a quadratic equation has a repeated root they can use the root twice to form a pair of identical factors and then form the quadratic expression.

(ii) Different Forms of a Quadratic Activity 1

The Workshop 7 Teacher Resources Booklet also contains an activity called "Different Forms of a Quadratic".

This document will use two forms from the booklet $y = x^2 - 4x - 5$ and $y = (x - 5)(x + 1)$

Graph both functions in the domain $-2 \leq x \leq 6$.



(a) What items of information from each of the forms can help us if sketching the graph of a function?

We can glean that the graph will be ∪-shaped and will have a y-intercept of –5 from $y = x^2 - 4x - 5$.

We can glean that the graph will have roots at $x = 5$ and $x = -1$ from $y = (x - 5)(x + 1)$. It is also easy enough to work out that the graph will be ∪-shaped.

(b) What algebraic skills are used to convert from one form to another?

We use the skills of expansion and factorising to transform from one form to the other.

(iii) Forming Quadratic Equations when Given Whole Number Roots

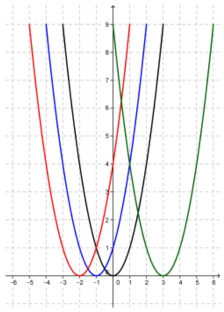
The language of Equation→ Factors → Roots was emphasised in a previous Unit in preparation for working from the roots to the equation. Topic 4.7 has the learning outcome "form quadratic equations given whole number roots".

$y = x^2 - 4x - 5$ and $y = (x - 5)(x + 1)$ were analysed above.

The roots of the function $y = x^2 - 4x - 5$ or $y = (x - 5)(x + 1)$ are $x = -1$ and $x = 5$.

The skill of using the roots of the function $x = -1$ and $x = 5$ to get the factors $(x + 1)$ and $(x - 5)$ to form the quadratic equation $(x - 5)(x + 1) = 0$ or $x^2 - 4x - 5 = 0$ could be dealt with now.

The functions in Set 5 above (x^2, $(x + 1)^2$, $(x + 2)^2$, $(x - 3)^2$ in the domains $-3 \leq x \leq 3, -4 \leq x \leq 2, -5 \leq x \leq 1, 0 \leq x \leq 6$) would be a good starting point for exploring questions where there is a repeated root.



Note: The activities above should make it a little easier to show the effect of "a" and "c" on $f(x) = ax^2 + bx + c$, especially with the aid of dynamic software. The curious student may ask about the effect of "b". b is the slope of the tangent at the y-intercept. The proof of this requires understanding of calculus.

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f'(0) = b$

Showing the slope of the tangent at the y-intercept for a few quadratic functions with different values for b using dynamic software should satisfy some of this student's curiosity.

Unit 22: The Difference of Two Squares

In this Unit students will:

- visualise the difference of two squares
- use information of functions of the form $f(x) = x^2 - a$ or $f(x) = (x + b)(x - b)$ to sketch the function.
- verbalise the effect of the "a" in $g(x) = x^2 - a$, using the language of transformation geometry
- verbalise the effect of the "b" in $g(x) = (x + b)(x - b)$, using the language of transformation geometry

The first three sections of this Unit will look at a variety of ways of investigating the difference of two squares. These are:

(i) The Difference of Two Squares from a Numerical Perspective,

(ii) The Difference of Two Squares from a Geometric Perspective and

(iii) The Difference of Two Squares from a Transformation of Functions Perspective.

The final section in the Unit looks at

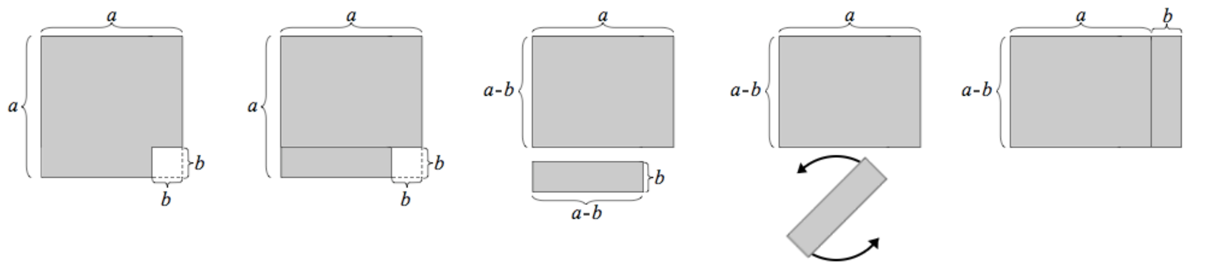
(iv) The Different Forms of a Quadratic Activity 2

<u>(i) The Difference of Two Squares from a Numerical Perspective</u>

Students could be asked to pick a (natural) number and find the product of the numbers that are one smaller and one larger than it. Record this product and compare it to the square of the original number. Ask them to verbalise what they see. For example, (9–1)(9+1)=(8)(10)=80 is one less than $(9)^2$. This is investigating $(x - 1)(x + 1) = x^2 - 1$. $(x - 2)(x + 2) = x^2 - 4$ and others of the form $(x + b)(x - b) = x^2 - b^2$ could also be investigated.

<u>(ii) The Difference of Two Squares from a Geometric Perspective</u>

One example of this is the work done [here for Reflections on Practice] where students cut different size squares from a 10×10 square. The area remaining can be made into a rectangle. The dimensions of which can be expressed as $10^2 - b^2 = (10 - b)(10 + b)$). The diagram below shows how this pattern can be generalised further to $a^2 - b^2 = (a - b)(a + b)$.



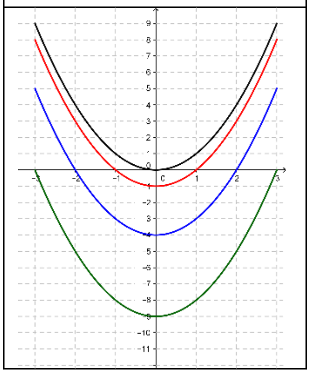
http://en.wikipedia.org/wiki/Difference_of_two_squares

(iii) The Difference of Two Squares from a Transformation of Functions Perspective

The pattern $(x - a)(x + a)$ could be explored from a transformation of functions point of view. Students investigated five sets of functions in the previous Unit and they could do something similar for a sixth set of functions in this Unit.

Set 6: $f(x) = x^2$, $g(x) = (x - 1)(x + 1)$, $h(x) = (x - 2)(x + 2)$, $p(x) = (x - 3)(x + 3)$ all in the domain $-3 \le x \le 3$.

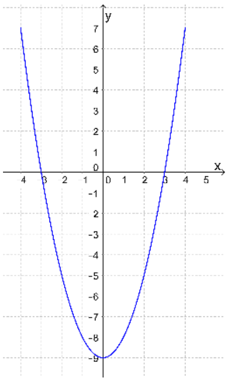
Set 6



(iv) The Different Forms of a Quadratic Activity 2

The previous Unit had a similar activity.

$y = x^2 - 9$ and $y = (x + 3)(x - 3)$ are two functions. Graph both functions in the domain $-4 \le x \le 4$.



(a) What items of information from each of the functions can help us if sketching the graph of a function?

We can glean that the graph will be U-shaped and will have a y-intercept of –9 from $y = x^2 - 9$.

We can glean that the function has roots at $x = -3$ and $x = +3$ and the graph will cut the x-axis at these values. From $y = (x + 3)(x - 3)$. It is also easy enough to work out that the graph will be U-shaped.

(b) What algebraic skills are used to convert from one form to another?

We use the skills of expansion and factorising to transform from one form to the other.

Note: Students will already be able to expand and simplify $(x + 3)(x - 3)$ into $x^2 - 9$. If the students haven't discovered the method for factorising $x^2 - 9$ from work done earlier in this Unit then now is a good time to do so. The diagram of $a^2 - b^2 = (a - b)(a + b)$ from earlier would be the most useful for this. Being able to express functions in factored form was very useful earlier for solving quadratic equations. The diagram of $a^2 - b^2 = (a - b)(a + b)$ transforms $a^2 - b^2$ into a rectangle and the area of a rectangle can be expressed as the product of two factors, which in this case would be $(a-b)(a+b)$.

Topic 4.6 has the learning outcome “factorise expressions such as….difference of two squares $a^2x^2 - b^2y^2$”. This type of question could be addressed now.

Unit 23: Cubic Expressions

In this Unit students will:

- evaluate expressions of the form $x^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbb{Z}$
- simplify expressions such as $ax(bx^2 + c)$ where $a, b, c \in \mathbb{Z}$
- multiply expressions of the form $(ax + b)(cx^2 + dx + e)$ where $a, b, c, d, e \in \mathbb{Z}$
- divide expressions of the form $(ax^3 + bx^2 + cx + d) \div (ex + f)$ where $a, b, c, d, e, f \in \mathbb{Z}$

Cubic functions are not listed in topic 5.2 of the Junior Cert. syllabus. If they were, it would be a good idea to look at their graphs. The graphs of cubic functions could still be explored. It would not take long to look at one cubic function even by just using ICT to see that some of the many things students learned about quadratic functions will also be true for functions of a higher order, for example, the link between linear factors and the roots of a function.

Students could be asked to substitute in values to expressions like $ax(bx^2 + c)$ and the simplified version of this i.e. $abx^3 + axc$. Students could evaluate expressions of the form $x^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbb{Z}$.

Earlier in this document students learned the skill of dividing a quadratic expression by a linear expression almost immediately after multiplying linear expressions by one another to get a quadratic expression. Students could be shown this work from a previous Unit. Something similar is suggested in this Unit for cubic expressions.

Students could be asked to do a small number of questions that require them to multiply a linear expression by a quadratic expression to get a product, which is cubic i.e. multiply expressions of the form $(ax + b)(cx^2 + dx + e)$ where $a, b, c, d, e \in \mathbb{Z}$. Then they could be asked to do a small number of questions that require them to do the reverse i.e. divide expressions of the form $(ax^3 + bx^2 + cx + d) \div (ex + f)$ where $a, b, c, d, e, f \in \mathbb{Z}$.

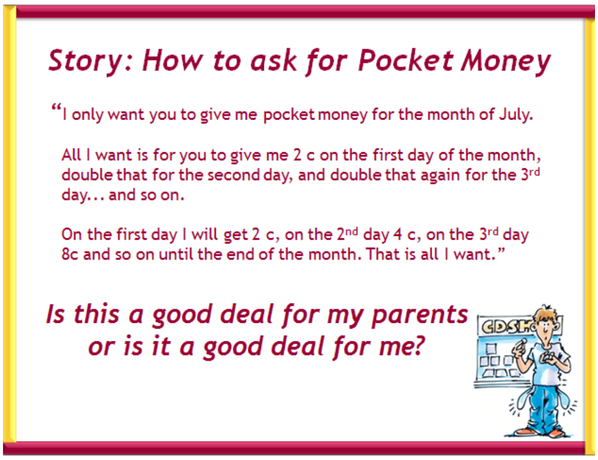
Note: The array model can be used for the multiplication and division skills mentioned above.

Unit 24: Exponential Patterns, Relationships and Functions

In this Unit students will:

- represent an exponential relationship in many ways
- identify the key features of functions $f(x) = 2^x$ and $g(x) = 3^x$
- identify the key features of functions $f(x) = a2^x$ and $g(x) = a3^x$
- find approximate solutions from graphs that show the comparison of two functions

The Pocket money question from Workshop 4 using multiple representations is good question for introducing exponential relations.



The 2^x, 3^x activity from Workshop 7 highlighted many of the key features of exponential functions. For both $f(x) = 2^x$ and $g(x) = 3^x$ and any function of the form $h(x) = a^x, a > 1$ the domain is $\mathbb{R}$, the range is $\mathbb{R}^+$ (or at junior cycle the positive real numbers), the y-intercept is 1, there is no x-intercept, the outputs of the functions are always positive and the function is ever increasing. By analysing the change in the values of the outputs it can be seen that the (average) change is always increasing when we look from low values to high values of x.

Functions of the form $f(x) = a2^x$ and $g(x) = a3^x$ can also be investigated.

Activities from Modular Course 3 that had different stimuli, for example, starting with a story, starting with a graph, starting with a table etc. could also be used.

Note: The shape of $f(x) = a2^x$ and $g(x) = a3^x$ can be connected to the shape of $100(1.04)^x$ when studying compound interest i.e. investing €100 at a rate of 4% per year compound interest.

Unit 25: Rearranging Formulae

In this Unit students will:

- rearrange formulae

Rearranging linear formulae was dealt at the time of studying other aspects of linear relationships. Another motivation for rearranging formulae could be to use the substitution method for simultaneous equations i.e. expressing each equation in terms of y and substituting.

In certain chapters, outside of algebra, it would be worth doing rearranging of the formula as it turns up in the topic i.e. rearranging when there is a need to rearrange. For example, when studying $A = l \times w$ if we know the length and width of a rectangle we can find out the area. If we know the area and the length what is the width? Can we rearrange the formula?

This Unit can be dealt with in 3rd year. Students could rearrange formulas they have rearranged before and also some new ones, possibly from the Formulas and Tables book.

Note: Other topics outside of algebra were mentioned above. If a formula can be expressed as a function then this could be shown to students, for example, *C*=2π*r* could be represented as *C*(*r*)= 2π*r*.

Unit 26: More Algebraic Fractions

In this Unit students will:

- add and subtract algebraic expressions such as $\frac{a}{bx+c} \pm \frac{p}{qx+r}$ where $a, b, c, p, q, r \in Z$

Unit 27: More Factorising by Grouping Questions e.g. $sx - ty + tx - sy$

In this Unit students will:

- factorise expressions such as $sx - ty + tx - sy$, where s, t, x, y are variable

Questions like factorising $pr + qr + ps + qs$ were addressed in an earlier Unit and each of the four terms were small rectangles that were all part of a larger rectangle. Factorising meant we could express the area of the rectangle concisely.

In this Unit students will factorise expressions like $sx - ty + tx - sy$. The array model can still be used.

$$\begin{aligned} &sx - ty + tx - sy \\ &sx + tx - sy - ty \\ &= x(s+t) - y(s+t) \\ &= (x-y)(s+t) \end{aligned}$$

	s	$+t$
x	sx	$+tx$
$-y$	s	$-sy$

$(x-y)(s+t)$

Note: It is possible to merge this Unit with the other Unit that included factorising by grouping.